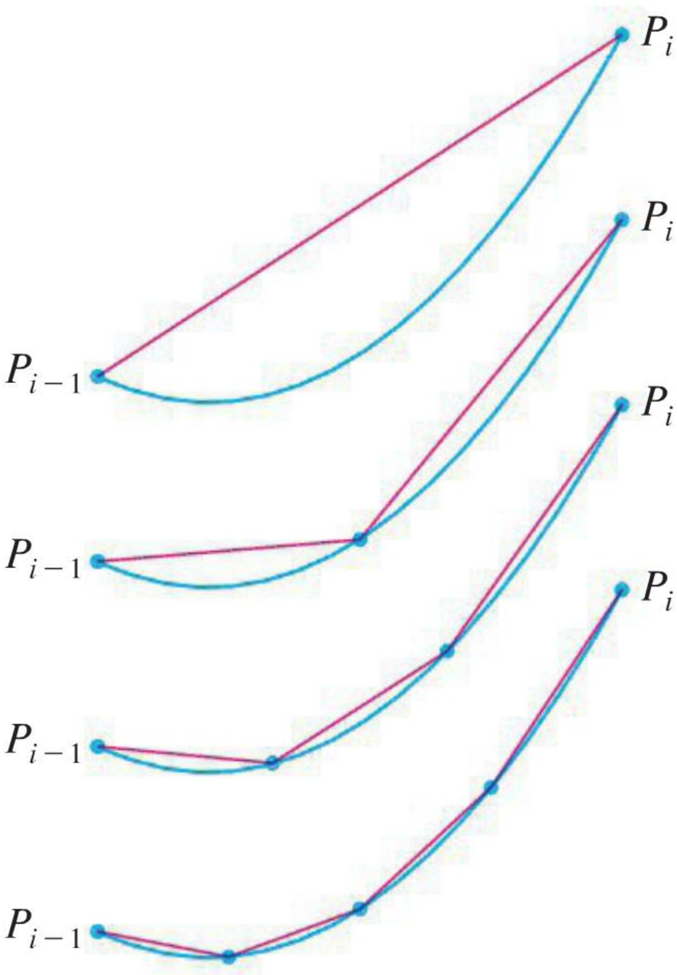
Sec. 13.3:
Arclength and Curvature

What We Will Go Over In Section 13.3

1. Arc Length
2. Different Parametrizations
3. Parametrizing With Respect to Arc Length
4. Curvature
5. The Normal and Binormal Vectors
6. The Normal and Osculating Planes

1. Arc Length

In Calc. II, arc length of a curve was defined as the limit of the total length of approximating polygons. Story…



If…

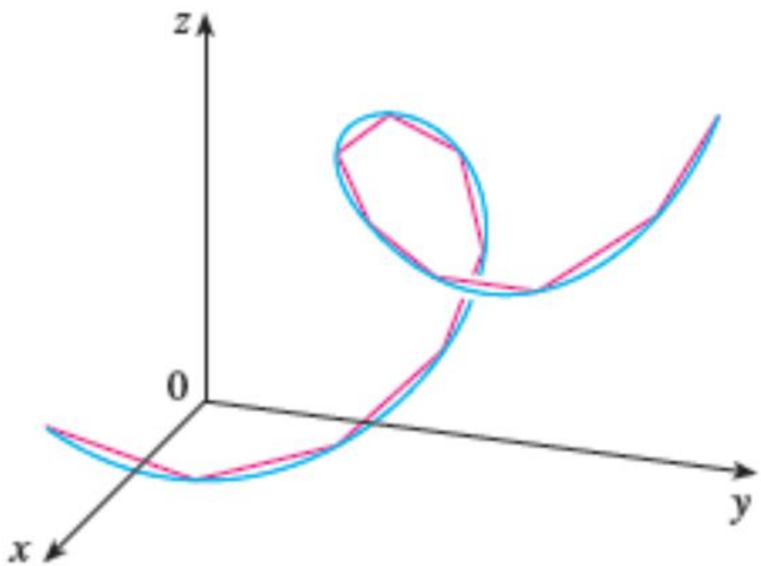
1) the curve is given parametrically,
2) $a < b$, and
3) the curve is traversed exactly once from $t = a$ to $t = b$,

then its arclength is …

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

1. Arc Length

In Calc. III, the arc length is still the limit of the total length of approximating polygons. The only difference is that the curve is a space curve.



If…

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2) $a < b$, and
3) the curve is traversed exactly once from $t = a$ to $t = b$,

then its arclength is …

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b |\vec{r}'(t)|\, dt$$

1. Arc Length

Ex 1: Find the length of the arc of the circular helix with vector equation
$\vec{r}(t) = < \cos t\ , \sin t\ ,\ \ t >$ from the point (1, 0, 0) to the point $(1, 0, 2\pi)$.

2. Different Parametrizations

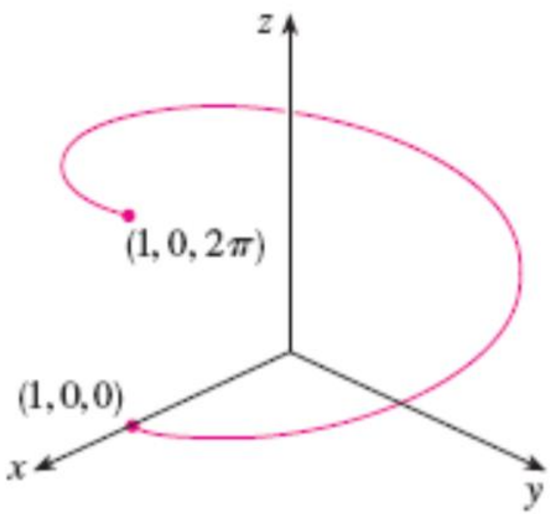
<u>Note</u>: Arc length does not depend on the parametrization of the curve.

<u>3 parametrizations of the same curve from example 1:</u>

$$\vec{r}(t) = < \cos t \ , \sin t \ , \ \ t > \qquad 0 \le t \le 2\pi$$

$$\vec{r}(u) = < \cos 2u \ , \sin 2u \ , \ \ 2u > \qquad 0 \le u \le \pi$$

$$\vec{r}(s) = < \cos(\ln s) \ , \sin(\ln s) \ , \ \ln s > \qquad 1 \le s \le e^{2\pi}$$



2. Different Parametrizations

<u>Note</u>: Arc length does not depend on the parametrization of the curve.

If…

1) $\vec{r}(t) = < x(t), y(t), z(t) >, \quad a \le t \le b$ is the arc of a space curve and

2) $t = f(s)$ is a monotonic function on $f^{-1}(a) \le s \le f^{-1}(b)$

Then…

3) $\vec{r}(s) = < x(f(s)), y(f(s)), z(f(s)) >,$
$f^{-1}(a) \le s \le f^{-1}(b)$ is a different parametrization of the arc of the same space curve and using the arc length formula will give you the same value for the arc length regardless of which parametrization you use.

3. Parametrizing With Respect to Arc Length

Normally we think of the parameter t as time. If an object is in motion along a space curve $\vec{r}(t)$, plugging in a value for t gives you the location of the object at that time.

Sometimes, we would like the input variable to be arc length (we will use s for this). If an object is in motion along a space curve $\vec{r}(s)$, plugging in a value for s gives you the location of the object after it has traveled a distance s along the curve from the starting point of the motion.

3. Parametrizing With Respect to Arc Length

The Arc Length Function

Let $\vec{r}(t) = < x(t), y(t), z(t) >, \quad a \le t \le b$ represent the arc of a space curve. The arclength function is the function $s(t)$ whose input is time and whose output is the length of the arc the object traveled from time a to time t.

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\; du$$

Other ways to write this…

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du \qquad\qquad \frac{ds}{dt} = |\vec{r}\,'(t)|$$

3. Parametrizing With Respect to Arc Length

<u>Parametrizing With Respect to Arc Length</u>

Let $\vec{r}(t) = < x(t), y(t), z(t) >, \quad a \leq t \leq b$ represent the arc of a space curve. Let $s(t)$ be the arclength function. Invert s (which means solve for t) and plug in to get $\vec{r}$ in terms of s. This is the parametrization of $\vec{r}$ with respect to arc length $\vec{r}$(s).

3. Parametrizing With Respect to Arc Length

Ex 2: Reparametrize the helix $\vec{r}(t) = < \cos t\,, \sin t\,, t >$ with respect to arc length measured from the point (1,0,0) in the direction of increasing t.

4. Curvature

Given a point on a space curve, its curvature κ at that point is a positive number that tells you how to what extent the curve is curving at that point.

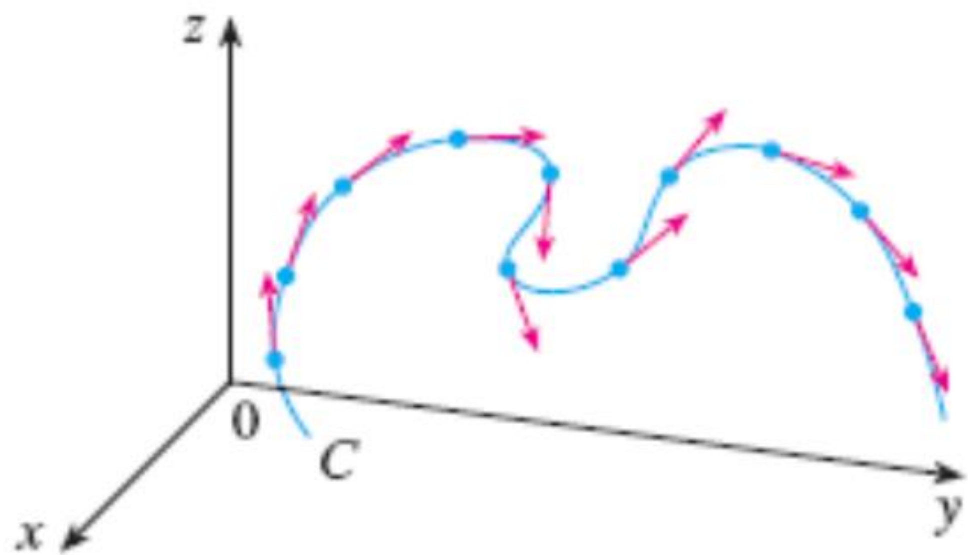
Story…

Draw some example on the board…

4. Curvature

<u>Curvature Idea</u>:

- It’s like the rate of change of the tangent vector
- But there are issues…
  1. Tangent vectors have differing lengths and will contribute to the rate of change. So instead of using $\vec{r}\,'(t)$, we’ll use the unit tangent vector $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

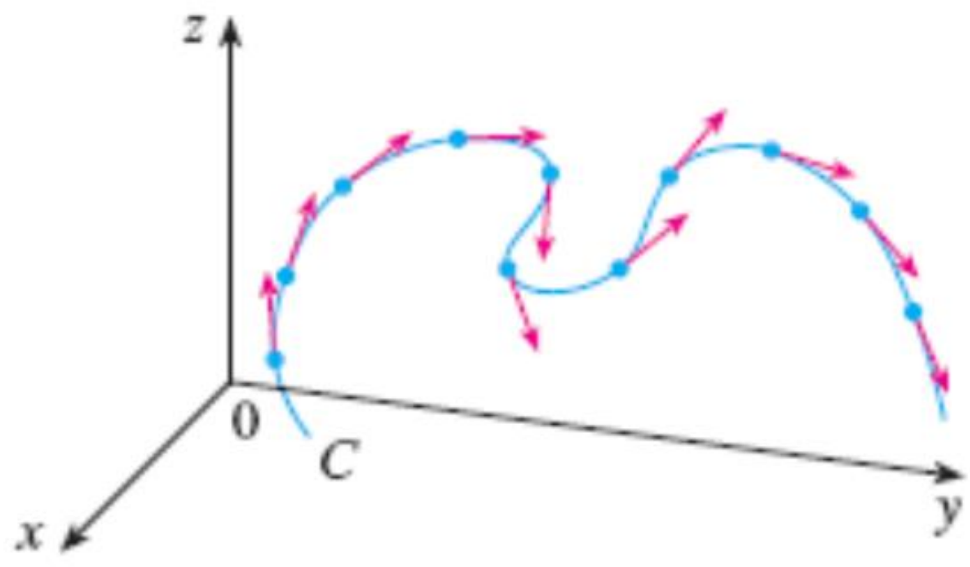


4. Curvature

<u>Curvature Idea</u>:

- It's like the rate of change of the tangent vector
- But there are issues…

  2. The derivative of the unit tangent vector depends on the parametrization used, so we will standardize it by using the parametrization with respect to arc length $\vec{T}(s)$.

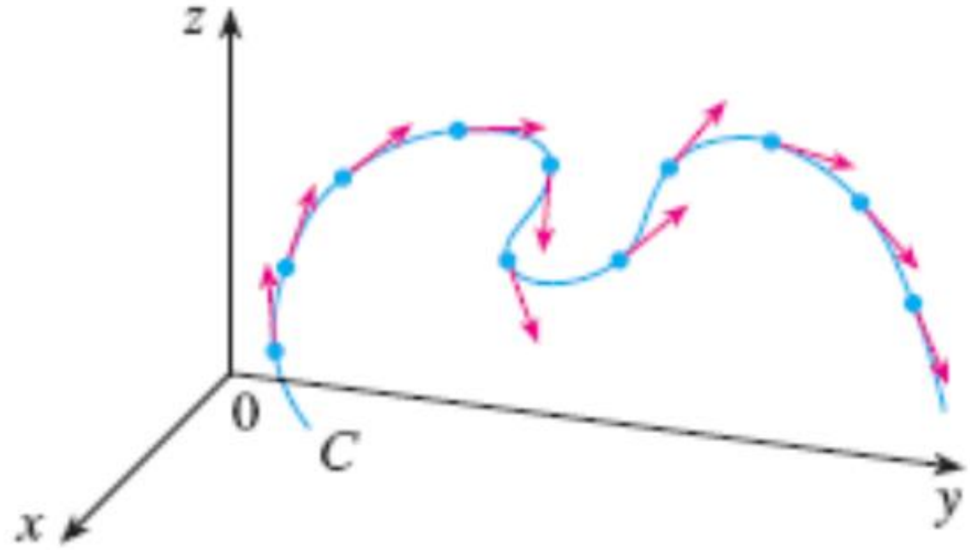


4. Curvature

Curvature Idea:

- It’s like the rate of change of the tangent vector
- But there are issues…

  3. Since curvature is a positive number, after we take the derivative, we will take its magnitude.



4. Curvature

<u>Definition of Curvature:</u>

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

This is the curvature function. To get the curvature at a specific point, plug in a number for s

Other formulas for curvature…

$$\kappa(t) = \frac{\left|\vec{T}'(t)\right|}{|\vec{r}'(t)|} \qquad \kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

If $r'(x) = < x, f(x) >$ $\rightarrow$ $$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

4. Curvature

<u>Ex 3</u>: Show that the curvature of a circle of radius a is $1/a$

4. Curvature

Ex 4: Find the curvature of the twisted cubic $\vec{r}(t) =< t, t^2, t^3 >$ at a general point and at (0,0,0).

4. Curvature

Ex 5: Find the curvature of the parabola $y = x^2$ at the points (0,0) , (1,1) , and (2,4).

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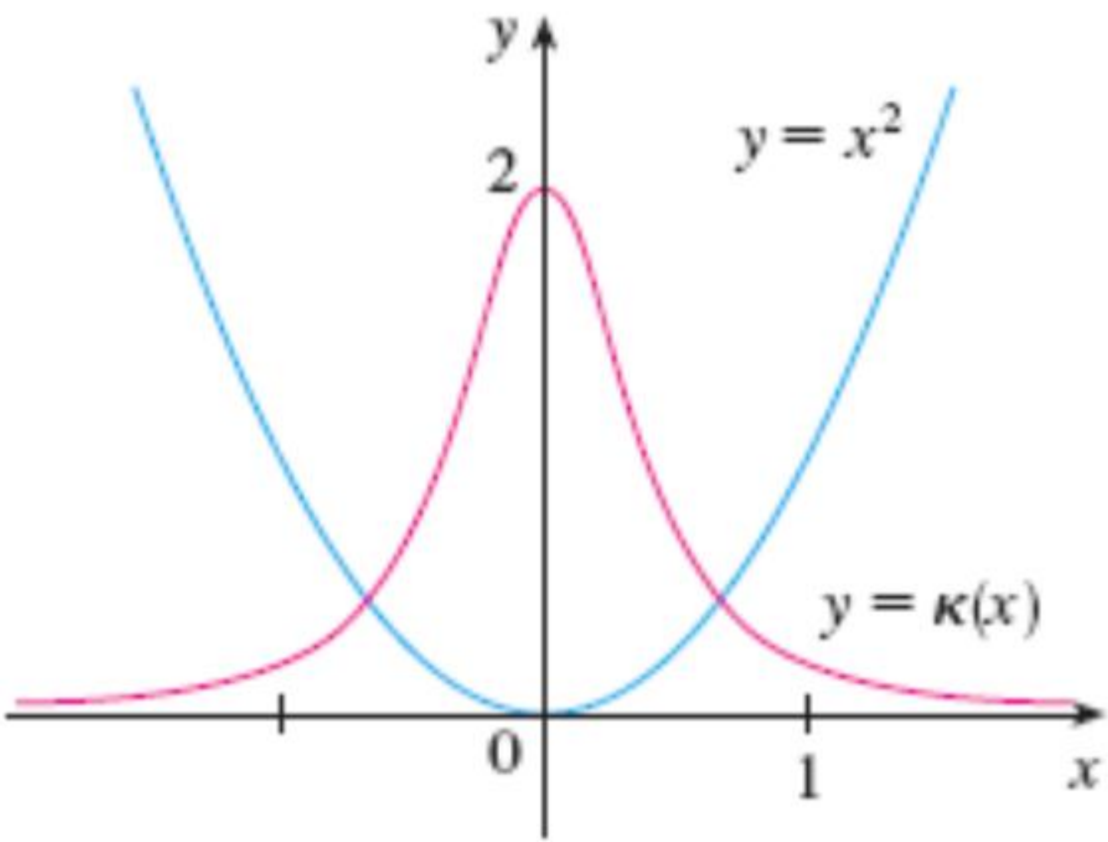


5. The Normal and Binormal Vectors

<u>The Unit Normal Vector</u>:

Given a space curve $\vec{r}(t)$ and a point P , consider the unit tangent vector $\vec{T}(t)$ at that point. There are many vectors perpendicular to $\vec{T}(t)$. One of them is called the <u>unit normal vector</u>

$$\vec{N}(t) \equiv \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$$

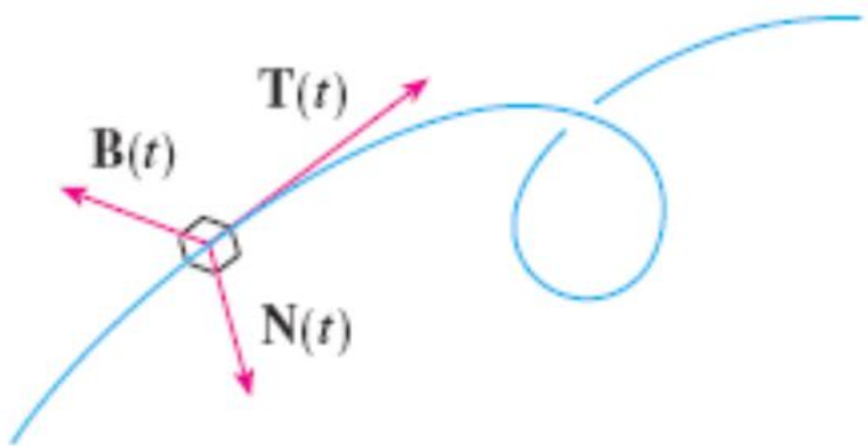
We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. Pic?

5. The Normal and Binormal Vectors

The Binormal vector:

Given a space curve $\vec{r}(t)$ and a point P , consider the unit tangent vector $\vec{T}(t)$ at that point. There are many vectors perpendicular to $\vec{T}(t)$. Another is called the binormal vector

$$\vec{B}(t) \equiv \vec{T}(t) \times \vec{N}(t)$$



5. The Normal and Binormal Vectors

Ex 6:

Find the unit normal and binormal vectors for the circular helix $\vec{r}(t) = <\cos t, \sin t, t>$

6. The Normal and Osculating Planes

Def:

1) The normal plane to $\vec{r}(t)$ at point P is the plane that passes through P and contains the unit normal and binormal vectors $\vec{N}(t)$ and $\vec{B}(t)$. So its normal vector is in the direction of $\vec{r}\,'(t)$ or $\vec{T}(t)$.
2) The osculating plane to $\vec{r}(t)$ at point P is the plane that passes through P and contains the unit tangent and unit normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. So its normal vector is in the direction of $\vec{B}(t)$.

6. The Normal and Osculating Planes

Def:

3) The osculating circle to $\vec{r}(t)$ at point P is the circle that

   a) lies in the osculating plane
   b) has the same tangent to C at P
   c) lies on the concave side of C (towards which $\vec{N}(t)$ points)
   d) and has radius $\rho = \frac{1}{\kappa}$ where κ is the curvature of C at P

6. The Normal and Osculating Planes

Ex 7: Find the equation of the normal plane and osculating plane of the helix from example 6 at the point $P(0, 1, \frac{\pi}{2})$

6. The Normal and Osculating Planes

Ex 8: Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

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